2006年9月 第29卷 第3期
湖南师范大学自然科学学报
Journal of Natural Science of Hunan Normal University
Vol.29 No.3 Sep.,2006

一个具最佳常数的双参数 Hardy-Hilbert 类不等式

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摘　要　引入双参数 λ_1,λ_2,利用权系数的方法,借助 Hölder 不等式,给出了一个具有最佳常数因子的积分型 Hardy-Hilbert 类不等式,作为应用,建立了它的等价形式及对应的二重级数不等式.

关键词　Hardy-Hilbert 类不等式;权系数;Hölder 不等式;最佳常数因子
中图分类号　O178　　**文献标识码**　A　　**文章编号**　1000-2537(2006)03-0005-04

A Hardy-Hilbert's Type Inequality with Two Parameter and the Best Constant Factor

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Abstract　By using the method of weight coefficient and Hölder's inequality and introducing two parameters λ_1, λ_2, and integral Hardy-Hilbert-type inequality with the best constant factor is given. As its application, the equivalent form and the corresponding double series inequalities are established.

Key words　Hardy-Hilbert's type inequality; Hölder's inequality; weight coefficient; the best constant factor

设 $p > 1, p^{-1} + q^{-1} = 1, f, g \geqslant 0, 0 < \int_0^\infty f^p(x)\mathrm{d}x < \infty, 0 < \int_0^\infty g^q(y)\mathrm{d}y < \infty$,则

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}\mathrm{d}x\mathrm{d}y < pq\left\{\int_0^\infty f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{1}$$

其对应的级数形式为

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{m=1}^\infty a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{m=1}^\infty b_n^q\right\}^{\frac{1}{q}}. \tag{2}$$

这里的常数因子 pq 为(1),(2) 的最佳值[1-5].

最近杨必成在文献[1] 中引入单参数 λ 将(1),(2) 进行了推广.

设 $p > 1, p^{-1} + q^{-1} = 1, f, g \geqslant 0, \lambda > 2 - \min\{p,q\}$,若 $0 < \int_0^\infty x^{1-\lambda}f^p(x)\mathrm{d}x < \infty, 0 < \int_0^\infty y^{1-\lambda}g^q(y)\mathrm{d}y < \infty$,则

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}\mathrm{d}x\mathrm{d}y < \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}\left\{\int_0^\infty x^{1-\lambda}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{1-\lambda}g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{3}$$

其等价不等式为

$$\int_0^\infty y^{(p-1)(\lambda-1)}\left[\int_0^\infty \frac{f(x)}{\max\{x^\lambda,y^\lambda\}}\mathrm{d}x\right]^p\mathrm{d}y < \left[\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}\right]^p\int_0^\infty x^{1-\lambda}f^p(x)\mathrm{d}x, \tag{4}$$

收稿日期:2006-03-01
基金项目:湖南省教育厅科研基金资助项目(03C440)

这里的常数因子$\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$,$\left[\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}\right]^p$分别是不等式(3),(4)的最佳值.其对应的级数形式为

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}<\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}\left\{\sum_{m=1}^{\infty}m^{1-\lambda}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}},\tag{5}$$

$$\sum_{n=1}^{\infty}n^{(p-1)(\lambda-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p<\left[\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}\right]^p\sum_{m=1}^{\infty}m^{1-\lambda}a_m^p,\tag{6}$$

常数因子$\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$,$\left[\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}\right]^p$也是不等式(5),(6)的最佳值.

本文引入双参数λ_1,λ_2,给出一个具有最佳常数因子的积分型Hardy-Hilbert类不等式,作为应用导出其等价式及相应的级数形式.

1 积分型Hilbert类不等式及其等价式

定理1 设$p>1,p^{-1}+q^{-1}=1,\lambda_1,\lambda_2>0,f(x),g(y)\geqslant 0$,若$0<\int_0^\infty x^{(p-1)(1-\lambda_1)}f^p(x)\mathrm{d}x<\infty$,$0<\int_0^\infty y^{(q-1)(1-\lambda_2)}g^q(y)\mathrm{d}y<\infty$,则

$$I=\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\mathrm{d}y<\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\int_0^\infty x^{(p-1)(1-\lambda_1)}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{(q-1)(1-\lambda_2)}g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}.\tag{7}$$

其等价形式为

$$\int_0^\infty y^{(\lambda_2-1)}\left[\int_0^\infty\frac{f(x)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\right]^p\mathrm{d}y<\left[\frac{pq}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right]\int_0^\infty x^{(p-1)(1-\lambda_1)}f^p(x)\mathrm{d}x,\tag{8}$$

这里的常数因子$\frac{pq}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$,$\left[\frac{pq}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\right]^p$分别是不等式(7),(8)的最佳值.

证 先证(7).定义权函数

$$\omega_{\lambda_1,\lambda_2}(x,q)=\int_0^\infty\frac{1}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\left[\frac{x^{\lambda_1+(1-\lambda_1)p}}{y^{\lambda_2+(1-\lambda_2)q}}\right]^{\frac{1}{q}}\mathrm{d}y;\omega_{\lambda_1,\lambda_2}(y,p)=\int_0^\infty\frac{1}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\left[\frac{y^{\lambda_2+(1-\lambda_2)q}}{x^{\lambda_1+(1-\lambda_1)p}}\right]^{\frac{1}{p}}\mathrm{d}x.$$

由带权的Hölder不等式有

$$\begin{aligned}I&=\int_0^\infty\int_0^\infty\left[\frac{f(x)}{(\max\{x^{\lambda_1},y^{\lambda_2}\})^{1/p}}\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)^{\frac{1}{pq}}\frac{y^{(\lambda_2-1)/p}}{y^{(\lambda_1-1)/q}}\right]\left[\frac{g(y)}{(\max\{x^{\lambda_1},y^{\lambda_2}\})^{1/q}}\left(\frac{y^{\lambda_2}}{x^{\lambda_1}}\right)^{\frac{1}{pq}}\frac{x^{(\lambda_1-1)/q}}{y^{(\lambda_2-1)/p}}\right]\mathrm{d}x\mathrm{d}y\leqslant\\&\left\{\int_0^\infty\int_0^\infty\frac{f^p(x)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)^{\frac{1}{q}}\frac{y^{(\lambda_2-1)}}{x^{(\lambda_1-1)p/q}}\mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{p}}\left\{\int_0^\infty\int_0^\infty\frac{g^q(y)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\left(\frac{y^{\lambda_2}}{x^{\lambda_1}}\right)^{\frac{1}{p}}\frac{x^{(\lambda_1-1)}}{y^{(\lambda_2-1)q/p}}\mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{q}}=\\&\left\{\int_0^\infty\omega_{\lambda_1,\lambda_2}(x,q)f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty\omega_{\lambda_1,\lambda_2}(y,p)g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}.\end{aligned}\tag{9}$$

若(9)取等号,则由文献[5],有$C\neq 0$,使$f^p(x)\left[\frac{x^{\lambda_1+(1-\lambda_1)p}}{y^{\lambda_2+(1-\lambda_2)q}}\right]^{\frac{1}{q}}=Cg^q(x)\left[\frac{y^{\lambda_2+(1-\lambda_2)q}}{x^{\lambda_1+(1-\lambda_1)p}}\right]^{\frac{1}{p}}$.即$x^{1+(p-1)(1-\lambda_1)}f^p(x)=Cy^{1+(q-1)(1-\lambda_2)}g^q(y)=B$,矛盾于$0<\int_0^\infty x^{(p-1)(1-\lambda_1)}f^p(x)\mathrm{d}x<\infty$.故有

$$I=\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\mathrm{d}y<\left\{\int_0^\infty\omega_{\lambda_1,\lambda_2}(x,q)f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty\omega_{\lambda_1,\lambda_2}(y,p)g^q(x)\mathrm{d}y\right\}^{\frac{1}{q}}.\tag{10}$$

而

$$\begin{aligned}\omega_{\lambda_1,\lambda_2}(x,q)&=\int_0^\infty\frac{1}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\left[\frac{x^{\lambda_1+(1-\lambda_1)p}}{y^{\lambda_2+(1-\lambda_2)q}}\right]^{\frac{1}{q}}\mathrm{d}y=\int_0^{x^{\lambda_1/\lambda_2}}\frac{1}{x^{\lambda_1}}\left[\frac{x^{\lambda_1+(1-\lambda_1)p}}{y^{\lambda_2+(1-\lambda_2)q}}\right]^{\frac{1}{q}}\mathrm{d}y+\\&\int_{x^{\lambda_1/\lambda_2}}^\infty\frac{1}{y^{\lambda_2}}\left[\frac{x^{\lambda_1+(1-\lambda_1)p}}{y^{\lambda_2+(1-\lambda_2)q}}\right]^{\frac{1}{q}}\mathrm{d}y=\frac{pq}{\lambda_2}x^{(p-1)(1-\lambda_1)}.\end{aligned}$$

同理可得:$\omega_{\lambda_1,\lambda_2}(y,p) = \int_0^\infty \frac{1}{\max\{x^{\lambda_1},y^{\lambda_1}\}}\left[\frac{y^{\lambda_2+(1-\lambda_2)q}}{x^{\lambda_1+(1-\lambda_1)p}}\right]^{\frac{1}{p}}\mathrm{d}x = \frac{pq}{\lambda_1}y^{(q-1)(1-\lambda_2)}$.代入(10)可得(7).

设 ε 为任意小的正数,$\bar{f}(x) = \left(\frac{1}{x}\right)^{[1+\lambda_1\varepsilon+(p-1)(1-\lambda_1)]/p}$,$\bar{g}(y) = \left(\frac{1}{y}\right)^{[1+\lambda_2\varepsilon+(q-1)(1-\lambda_2)]/q}$,则

$$\int_0^\infty\int_0^\infty \frac{\bar{f}(x)\bar{g}(y)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\mathrm{d}y = \int_0^\infty\left(\frac{1}{x}\right)^{[1+\lambda_1\varepsilon+(p-1)(1-\lambda_1)]/p}\left[\int_0^\infty \frac{1}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\left(\frac{1}{y}\right)^{[1+\lambda_2\varepsilon+(q-1)(1-\lambda_2)]/q}\mathrm{d}y\right]\mathrm{d}x =$$

$$\frac{q^2}{\lambda_2(q-1-\varepsilon)(1+\varepsilon)}\int_0^\infty\left(\frac{1}{x}\right)^{\frac{1+\lambda_1\varepsilon+(p-1)(1-\lambda_1)}{p}+\frac{\lambda_1(1+\varepsilon)}{q}}\mathrm{d}x = \frac{q^2}{\lambda_2(q-1-\varepsilon)(1+\varepsilon)}\int_0^\infty\left(\frac{1}{x}\right)^{1+\lambda_1\varepsilon}\mathrm{d}x =$$

$$\frac{q^2}{\lambda_1\lambda_2(q-1-\varepsilon)(1+\varepsilon)\varepsilon},\tag{11}$$

$$\left[\int_0^\infty x^{(p-1)(1-\lambda_1)}\bar{f}^p(x)\mathrm{d}x\right]^{\frac{1}{p}}\left[\int_0^\infty y^{(q-1)(1-\lambda_2)}\bar{g}^q(y)\mathrm{d}y\right]^{\frac{1}{q}} = \frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}.\tag{12}$$

若(7)的常数因子不为最佳值,则存在正数 $K < \frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}\varepsilon}$,使(7)的常数因子换成 K 后仍成立.于是由(11),(12)可得不等式$\frac{q^2}{\lambda_1\lambda_2(q-1-\varepsilon)(1+\varepsilon)} < K < \frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}}$,即$\frac{q^2}{\lambda_1^{1/q}\lambda_2^{1/p}(q-1)} = \frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}} \leqslant K(\varepsilon \to 0^+)$.这与 $K < \frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}$ 矛盾,所以(7)的常数因子为最佳值.

再证(8).设 $g(y) = y^{(\lambda_2-1)}\left[\int_0^\infty \frac{f(x)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\right]^{p-1}$,由(7)有

$$\int_0^\infty y^{(q-1)(1-\lambda_2)}g^q(y)\mathrm{d}y = \int_0^\infty y^{\lambda_2-1}\left[\int_0^\infty \frac{f(x)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\right]^p\mathrm{d}y = \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\mathrm{d}y <$$

$$\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\int_0^\infty x^{(p-1)(1-\lambda_1)}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{(q-1)(1-\lambda_2)}g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}.$$

由此有

$$\left[\int_0^\infty y^{(q-1)(1-\lambda_2)}g^q(y)\mathrm{d}y\right]^{\frac{1}{p}} < \frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\int_0^\infty x^{(p-1)(1-\lambda_1)}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}},$$

即

$$\int_0^\infty y^{(\lambda_2-1)}\left[\int_0^\infty \frac{f(x)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\right]^p\mathrm{d}y < \left[\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p\int_0^\infty x^{(p-1)(1-\lambda_1)}f^p(x)\mathrm{d}x.$$

故不等式(8)成立.

最后证(7)与(8)等价.从以上证明可知由(7)能得到(8).反之由 Hölder 不等式有

$$I = \int_0^\infty\left[y^{(q-1)(\lambda_2-1)/q}\int_0^\infty \frac{f(x)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\right][y^{(1-\lambda_2)(q-1)/q}g(y)]\mathrm{d}y \leqslant$$

$$\left\{\int_0^\infty y^{\lambda_2-1}\left[\int_0^\infty \frac{f(x)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}\mathrm{d}x\right]^p\mathrm{d}y\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{(q-1)(1-\lambda_2)}g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}.\tag{13}$$

在(13)中应用(8)即可得(7).因此(7)和(8)等价.若(8)的常数因子不是最佳值,则由(13)得到的(7)的常数因子也不是最佳值,这与前面的证明矛盾.故常数因子$\left[\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p$是(8)的最佳值,证毕.

2　推广的级数形式

定理 2　设 $p > 1, p^{-1} + q^{-1} = 1, \lambda_1, \lambda_2 > 0, a_m, b_n \geqslant 0$, 若 $0 < \sum_{m=1}^\infty m^{(p-1)(1-\lambda_1)}a_m^p < \infty, 0 < \sum_{m=1}^\infty n^{(q-1)(1-\lambda_2)}b_n^q < \infty$,则

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda_1},n^{\lambda_2}\}} < \frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\sum_{m=1}^{\infty}m^{(p-1)(1-\lambda_1)}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda_2)}b_n^q\right\}^{\frac{1}{q}}, \tag{14}$$

$$\sum_{n=1}^{\infty}n^{\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^{\lambda_1},n^{\lambda_2}\}}\right]^p < \left[\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p\sum_{m=1}^{\infty}m^{(p-1)(1-\lambda_1)}a_m^p, \tag{15}$$

这里的常数因子$\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}$, $\left[\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p$ 分别是不等式(14),(15) 的最佳值.

证 权系数 $\overline{\omega}_{\lambda_1,\lambda_2}(m,q) = \sum_{n=1}^{\infty}\frac{1}{\max\{m^{\lambda_1},n^{\lambda_2}\}}\left[\frac{m^{\lambda_1+(1-\lambda_1)p}}{n^{\lambda_2+(1-\lambda_2)q}}\right]^{\frac{1}{q}} <$

$$\omega_{\lambda_1,\lambda_2}(m,q) = \int_0^{\infty}\frac{1}{\max\{m^{\lambda_1},y^{\lambda_2}\}}\left[\frac{m^{\lambda_1+(1-\lambda_1)p}}{y^{\lambda_2+(1-\lambda_2)q}}\right]^{\frac{1}{q}}\mathrm{d}y = \frac{pq}{\lambda_2}m^{(p-1)(1-\lambda_1)}.$$

$$\overline{\omega}_{\lambda_1,\lambda_2}(n,p) = \sum_{m=1}^{\infty}\frac{1}{\max\{m^{\lambda_1},n^{\lambda_2}\}}\left[\frac{n^{\lambda_2+(1-\lambda_2)q}}{m^{\lambda_1+(1-\lambda_1)p}}\right]^{\frac{1}{p}} <$$

$$\omega_{\lambda_1,\lambda_2}(n,p) = \int_0^{\infty}\frac{1}{\max\{x^{\lambda_1},n^{\lambda_2}\}}\left[\frac{n^{\lambda_2+(1-\lambda_2)q}}{x^{\lambda_1+(1-\lambda_1)p}}\right]^{\frac{1}{p}}\mathrm{d}x = \frac{pq}{\lambda_1}n^{(q-1)(1-\lambda_2)}.$$

类似于定理 1 的证明,由 Hölder 不等式有

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda_1},n^{\lambda_2}\}} < \frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\sum_{m=1}^{\infty}m^{(p-1)(1-\lambda_1)}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda_2)}b_n^q\right\}^{\frac{1}{q}}.$$

设 ε 为任意小的正数, $\overline{a_m} = \left(\frac{1}{m}\right)^{[1+\lambda_1\varepsilon+(p-1)(1-\lambda_1)]/p}$, $\overline{b_n} = \left(\frac{1}{n}\right)^{[1+\lambda_2\varepsilon+(q-1)(1-\lambda_2)]/q}$. 则有

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\overline{a_m}\cdot\overline{b_n}}{\max\{m^{\lambda_1},n^{\lambda_2}\}} > \int_0^{\infty}\int_0^{\infty}\frac{\overline{f}(x)\overline{g}(y)}{\max\{x^{\lambda_1},n^{\lambda_2}\}}\mathrm{d}x\mathrm{d}y = \frac{q^2}{\lambda_1\lambda_2(q-1-\varepsilon)(1+\varepsilon)\varepsilon}. \tag{16}$$

因为$\sum_{m=1}^{\infty}m^{(p-1)(1-\lambda_1)}\overline{a}_m^p = \sum_{m=1}^{\infty}\frac{1}{m^{1+\lambda_1\varepsilon}} = \frac{1}{\lambda_1\varepsilon} + O(1)$,所以有:

$$\left\{\sum_{m=1}^{\infty}m^{(p-1)(1-\lambda_1)}\overline{a}_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}m^{(q-1)(1-\lambda_2)}\overline{b}_n^p\right\}^{\frac{1}{q}} =$$

$$\left[\frac{1}{\lambda_1\varepsilon} + O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\lambda_2\varepsilon} + O(1)\right]^{\frac{1}{q}} = \frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}(1 + O(1)).$$

由(16) 和上式易证(14) 的常数因子是最佳值.

设 $b_n = n^{(\lambda_2-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^{\lambda_1},n^{\lambda_2}\}}\right]^{p-1}$,则由(14) 有

$$\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda_2)}\overline{b_n^q} = \sum_{n=1}^{\infty}n^{(\lambda_2-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^{\lambda_1},n^{\lambda_2}\}}\right]^p = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda_1},n^{\lambda_2}\}} <$$

$$\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{\sum_{m=1}^{\infty}m^{(p-1)(1-\lambda_1)}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda_2)}b_m^q\right\}^{\frac{1}{q}},$$

由此可得

$$\sum_{n=1}^{\infty}n^{\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^{\lambda_1},n^{\lambda_2}\}}\right]^p < \left[\frac{pq}{\lambda_1^{1/q}\lambda_2^{1/p}}\right]^p\sum_{m=1}^{\infty}m^{(p-1)(1-\lambda_1)}a_m^p.$$ 故不等式(15) 成立.

类似于定理 1 可证(14)和(15)等价,及(15)的常数因子为最佳值,证毕.

参考文献:

[1] 杨必成.一个推广的 Hilbert 类不等式及应用[J].工程数学学报,2004,5:821-824.

[2] HARDY G H,LITTLEWOOD J E,POLYA G.Inequalities[M].Cambridge:Cambridge Unive Press,1952.

[3] 杨必成.一个推广的具有最佳常数的 Hardy-Hilbert 积分不等式[J].数学年刊,2002,2(4):401-408.

[4] KUANG J C,DEBNATH L.On new generalizations of Hilbert's inequality and their applications[J]. J Math Anal Appl,2000,245:248-265.

[5] 匡继昌.常用不等式[M].长沙:湖南教育出版社,1993.